A more accurate treatment of the problem of drawing the shortest line on a surface*

Leonhard Euler

1. For a surface on which a shortest line is to be drawn, given this differential equation between the three orthogonal coordinates x, y, z: $dz = fdx + gdy$, where f and g are functions of both x and y, let it be

$$df = \alpha dx + \beta dy \quad \text{and} \quad dg = \beta dx + \gamma dy.$$

Having done this, since an element of any line drawn on this surface would be

$$\sqrt{dx^2 + dy^2 + dz^2},$$

with the previous value put in place of dz an element of this curve will be

$$= \sqrt{dx^2 + dy^2 + (fdx + gdy)^2};$$

whence if we put $dy = pdx$, this element will be $dx\sqrt{1 + pp + (f + gp)^2}$.

2. This integral formula, which is to be brought to a minimum, will be

$$\int dx\sqrt{1 + pp + (f + gp)^2},$$

which I indicated in general by $\int Zdx$ in my Treatise *Methodus inveniendi lineas curvas Maximi Minimive proprietate gaudentes*, so that in this case it would be

$$Z = \sqrt{1 + pp + (f + gp)^2}.$$

Then, having put $dZ = Mdx + Ndy + Pdp$, I have shown that the nature of a Minimum or a Maximum is such that it is expressed by this equation:[1] $Ndx = dP$, which clearly leads to differentials of the second order.

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[1]Translator: This is namely the Euler-Lagrange equation in the calculus of variations.

3. Therefore since $Z^2 = 1 + pp + (f + gp)^2$, let us differentiate this formula and let the elements be separated into the three types, namely dx, dy, dp, and then be represented in this way:

$$ZdZ = dx(\alpha + \beta p)(f + gp) + dy(\beta + \gamma p)(f + gp) + dp(p + g(f + gp)).$$

Then since I have put in general $dZ = Mdx + Ndy + Pdp$, we will have in this case:

$$\begin{aligned} M &= \frac{(\alpha + \beta p)(f + gp)}{Z} \\ N &= \frac{(\beta + \gamma p)(f + gp)}{Z} \\ P &= \frac{p + g(f + gp)}{Z}. \end{aligned}$$

Hence (since $\beta dx + \gamma p dx = dg$) it becomes

$$Ndx = \frac{dg(f + gp)}{Z},$$

whence the equation for our sought curve will be

$$\frac{dg(f + gp)}{Z} = d \cdot \frac{p + g(f + gp)}{Z}.$$

For expanding this equation, for the sake of brevity let us put $p + g(f + gp) = S$, and we will have:

$$\frac{dg(f + gp)}{Z} = \frac{dS}{Z} - \frac{SdZ}{ZZ}$$

or

$$dg(f + gp) = dS - \frac{SdZ}{Z}.$$

Therefore because $dS = dp + dg(f + gp) + gd(f + gp)$, our equation will be

$$0 = dp + gd(f + gp) - \frac{SdZ}{Z}.$$

But on the other hand it is:

$$\frac{dZ}{Z} = \frac{pdp + (f + gp)d(f + gp)}{1 + pp + (f + gp)^2},$$

which should be multiplied by $S = p + g(f + gp)$. Then multiplying by the denominator $1 + pp + (f + gp)^2$ we will have:

$$0 = dp + (g - fp)d(f + gp) - gpdp(f + gp) + dp(f + gp)^2$$

or

$$0 = dp + (g - fp)d(f + gp) + fdp(f + gp),$$

which we can then write in this form:

$$0 = dp(1 + ff + gg) + (g - fp)(df + pdg).$$

4. Although this equation is simple enough, it is not clear however how it could be reduced to differentials of the first degree. But I have observed that the problem can be dealt with by the following substitution, namely: $v = \frac{g-fp}{f+gp}$; whence it will be $p = \frac{g-fv}{gv+f}$, and now by differentiating we deduce

$$dp = \frac{-(ff + gg)dv + (1 + vv)(fdg - gdf)}{(f + gv)^2}.$$

Further, it will be

$$g - fp = \frac{v(ff + gg)}{f + gv},$$

and next

$$df + pdg = \frac{fdf + gdg + v(gdf - fdg)}{f + gv}.$$

With these substituted in, the equation becomes:

$$\begin{aligned} 0 \quad = \quad & -dv(ff + gg)(1 + ff + gg) + v(ff + gg)(fdf + gdg) \\ & +(1 + vv)(fdg - gdf) + (ff + gg)(fdg - gdf). \end{aligned}$$

5. In order to turn this into a simpler equation let us set $ff + gg = hh$, and it will be $fdf + gdg = hdh$, and next let $\frac{g}{f} = k$, so that it would become $fdg - gdf = ffdk$, and thus our equation is contracted into this form:

$$0 = -hhdv(1 + hh) + h^3vdh + (1 + hh + vv)ffdk.$$

Also since $g = fk$, it will be $ff(1 + kk) = hh$ and then $ff = \frac{hh}{1+kk}$, from which we will have:

$$0 = -dv(1 + hh) + vhdh + (1 + hh + vv)\frac{dk}{1 + kk};$$

next by putting $v = s\sqrt{1 + hh}$ this equation is reduced to this form:

$$0 = -ds\sqrt{(1 + hh)} + \frac{dk(1 + ss)}{1 + kk}.$$

Therefore now the quantity s can be exhibited separately from the others, as

$$\frac{ds}{1 + ss} = \frac{dk}{(1 + kk)\sqrt{1 + hh}},$$

which seems to be the simplest form to which it can be brought in general.

6. While we have attempted to determine the two variables y and x by the single variable z, since all three are involved in equal measure in the calculation, it appeared to me to treat this whole question so that all the formulae involve in equal measure the three coordinates x, y, z; this idea will be put to good use in the following investigations that I will take up.

Supplement

7. Let the differential equation of a given surface be: $pdx + qdy + rdz = 0$, where p, q, r are functions of the coordinates x, y, z: whence for this equation to be possible, this condition must be satisfied:[2]

$$\frac{pdq - qdp}{dz} + \frac{qdr - rdq}{dx} + \frac{rdp - pdr}{dy} = 0.$$

With this done, the following equation for drawing the shortest line on this surface will be obtained, which involves the three coordinates x, y, z is equal measure:

$$ddx(qdz - rdy) + ddy(rdx - pdz) + ddz(pdy - qdx) = 0.$$

Or if we put for the sake of brevity:

$$\begin{aligned} dyddz - dzddy &= f, \\ dzddx - dxddz &= g, \\ dxddy - dyddx &= h, \end{aligned}$$

it will be $fp + gq + hr = 0$, then indeed on the other hand $fdx + gdy + hdz = 0$. Next if an element of the shortest curve is put $= ds$, it will be $ds^2 = dx^2 + dy^2 + dz^2$, then indeed also

$$\frac{dds}{ds} = \frac{qddz - rddy}{qdz - rdy} = \frac{rddx - pddz}{rdx - pdz} = \frac{pddy - qddx}{pdy - qdx}.$$

Application to a spherical surface

8. Let the equation for the surface be $xdx + ydy + zdz = 0$, so that we have here $p = x, q = y, r = z$, and the first equation for the shortest path will be the following:

$$ddx(ydz - zdy) + ddy(zdx - xdz) + ddz(xdy - ydx) = 0,$$

whose complete integral is thus $\alpha x + \beta y + \gamma z = 0$, which is apparent from the nature of the matter. The question is therefore reduced to how this integral can be worked out.

[2]Translator: cf. p. 101 of D. J. Struik, *Outline of a history of differential geometry: I*, Isis **19** (1933), no. 1, 92–120

9. Now with the earlier equation $fx+gy+hz=0$, if for this equation we put $\Pi = \frac{zdx-xdz}{ydx-xdy}$, it will be $d\Pi = d\cdot\frac{zdx-xdz}{ydx-xdy}$ and then

$$d\Pi = \frac{zddx-xddz}{ydx-xdy} - \frac{(zdx-xdz)(yddx-xddy)}{(ydx-xdy)^2},$$

or by expanding

$$d\Pi = \frac{x}{(ydx-xdy)^2}((dyddz-dzddy)x+(dzddx-dxddz)y+(dxddy-dyddx)z)$$

and with the f,g,h that have been introduced it will be

$$d\Pi = x\frac{(fx+gy+hz)}{(ydx-xdy)^2}.$$

Since on the other hand it is $fx+gy+hz=0$, it will be $d\Pi=0$ and hence Π is a constant quantity, which if we put $=A$, the differential equation of the first degree $\Pi=\frac{zdx-xdz}{ydx-xdy}$ can be thus expressed:

$$A(ydx-xdy) = zdx-xdz,$$

which when divided by xx will be integrable; for it would become

$$\frac{Ay}{x} = \frac{z}{x}+B \quad \text{or} \quad Ay-Bx-z=0$$

or with the constants switched

$$\alpha x+\beta y+\gamma z=0.$$

Since this equation is clearly everywhere drawn from the center of the sphere, great circles arise on the surface of the sphere; whence it follows that all great circles are all the shortest paths which can be drawn on the surface of the sphere.

10. Since in these calculations everything is typically reduced to a single variable, if for effecting this we put $dy=tdx$ and $dx=udx$, taking dx as constant, the first equation will be as follows:

$$dt(r-pu)+du(pt-q)=0.$$

And the equation for the surface will be $p+qt+ru=0$; whence, since it then becomes $p=-qt-ru$, the former equation takes this form:

$$dt(r+qtu+ruu)-du(q+rtu+qtt)=0.$$

Next it will be

$$f=dx^2(tdu-udt), \quad g=-dx^2du, \quad h=dx^2dt,$$

then indeed $ds^2 = dx^2(1 + tt + uu)$ and finally

$$\frac{dds}{ds} = \frac{tdt + udu}{1 + tt + uu} = \frac{qdu - rdt}{qu - rt} = -\frac{pdu}{r - pu} = \frac{pdt}{pt - q}.$$

11. And if desired to introduce as a fourth variable the angle φ, by putting $dx = td\varphi, dy = ud\varphi, dz = vd\varphi$, the equation for the surface will be

$$pt + qu + rv = 0.$$

Next for the letters f, g, h we will have

$$\begin{aligned} f &= d\varphi^2(udv - vdu), \\ g &= d\varphi^2(vdt - tdv), \\ h &= d\varphi^2(tdu - udt), \end{aligned}$$

thus it will then be $ft + gu + hv = 0$. The equation for the shortest path will be:

$$fp + gq + hr = p(udv - vdu) + q(vdt - tdv) + r(tdu - udt) = 0,$$

and it would finally become $ds^2 = d\varphi^2(tt + uu + vv)$ and hence

$$\frac{dds}{ds} = \frac{tdt + udu + vdv}{tt + uu + vv} = \frac{qdv - rdu}{qv - ru} = \frac{rdt - pdv}{rt - pv} = \frac{pdu - qdt}{pu - qt}.$$

12. Since no way presents itself to us for integrating the general equation given above for drawing the shortest path on a surface, even though many cases can be given in which integration of the equation for the curve succeeds, it will be worthwhile to have expanded several of these here as a conclusion.

13. Let us begin with the case in which one of the quantities p, q, r vanishes. In particular, if $r = 0$, then the equation for the surface will be $pdx + qdy = 0$, in which case therefore the surface is a cylinder, whose base is determined by the equation $pdx + qdy$. And putting in the equation for $\frac{dds}{ds}$ the given $r = 0$, it would become $\frac{dds}{ds} = \frac{ddz}{dz}$, whose integral is $lds = ldz + l\alpha$ and hence, taking numbers, $ds = \alpha dz$ or

$$dx^2 + dy^2 + dz^2 = \alpha\alpha dz^2 \quad \text{or} \quad dx^2 + dy^2 = dz^2(\alpha\alpha - 1).$$

It will then be

$$z\sqrt{\alpha\alpha - 1} = \int \sqrt{dx^2 + dy^2},$$

where $\int \sqrt{dx^2 + dy^2}$ expresses an element of the base of the curve; whence it is apparent that the altitude z of a cylinder is always proportional to the arc of the base.

14. Let us consider the case in which $p = x$ and $q = y$, where the equation for the surface will thus be:

$$xdx + ydx + rdz = 0,$$

in which all spherical bodies or rotated surfaces are contained. Then it will further be

$$\frac{dds}{ds} = \frac{xddy - yddx}{xdy - ydx},$$

whose integral is

$$lds = l(xdy - ydx) + la,$$

and hence by taking numbers

$$\frac{ds}{a} = xdy - ydx \quad \text{and so} \quad \frac{dx^2 + dy^2 + dz^2}{aa} = (xdy - ydx)^2$$

or switching the constants

$$(dx^2 + dy^2 + dz^2)AA = (xdy - ydx)^2.$$

For integrating this equation again let us put $x = v\cos\varphi$ and $y = v\sin\varphi$, and it will be

$$dx^2 + dy^2 = dv^2 + vvd\varphi^2;$$

then indeed for the curve it will be $vdv + rdz = 0$, where r is some function of v, so that $dz = -\frac{vdv}{r}$. Next indeed it will be

$$xdy - ydx = vvd\varphi;$$

and with all these substitutions our equation will be:

$$AA(dv^2 + vvd\varphi^2 + \frac{vvdv^2}{rr}) = AAds^2 = v^4 d\varphi^2,$$

from which in turn we gather

$$d\varphi^2 = \frac{AA(dv^2 + \frac{vvdv^2}{rr})}{v^4 - vvAA} = \frac{AAdv^2(rr + vv)}{rrvv(vv - AA)}$$

and hence

$$d\varphi = \frac{Adv}{rv}\sqrt{\frac{rr + vv}{vv - AA}}.$$

15. For other cases our general equation above can be put to more use. And first indeed, because so far things have depended on a ratio between the quantities p, q, r, one of which will be assumed at our pleasure. Thus let $r = -1$, so it becomes $dz = pdx + qdy$, and let p and q be functions of x and y with it

being $(\frac{dp}{dy}) = (\frac{dq}{dz})$. Next let us put $dy = \pi dx$, and it will be $dz = (p + \pi q)dx$. Then, taking the element dx as constant, so that it would be $ddx = 0$, it will be

$$ddy = d\pi dx \quad \text{and} \quad ddz = (dp + \pi dq + qd\pi)dx.$$

Now the three letters f, g, h can be expressed in the following way:

$$\begin{aligned} f &= dx^2(\pi dp - pd\pi + \pi\pi dq), \\ g &= -dx^2(dp + \pi dq + qd\pi), \\ h &= d\pi dx^2. \end{aligned}$$

Indeed we have seen that the equation for the shortest path is

$$pf + gq + hr = 0,$$

which therefore takes this form:

$$-d\pi(1 + pp + qq) + dp(\pi p - q) + \pi dq(\pi p - q) = 0$$

or

$$d\pi(1 + pp + qq) + (dp + \pi dq)(q - \pi p) = 0.$$

16. Since the two formulae $p + \pi q$ adn $q - \pi p$ are the central components of this equation, it will be very helpful to work out a relation between them. To this end let us put $\frac{q-\pi p}{p+\pi q} = v$, whence it is now $\pi = \frac{q-vp}{p+vq}$; then indeed in turn

$$g - \pi p = \frac{v(pp + qq)}{p + vq},$$

and on the other hand it will be

$$dp + \pi dq = \frac{pdp + qdq + v(qdp - pdq)}{p + qv}.$$

Now if we put $q = up$, it will be

$$\pi = \frac{u - v}{1 + uv} \quad \text{and then} \quad d\pi = \frac{du(1 + vv) - dv(1 + uu)}{(1 + uv)^2}.$$

Next let us put $pp + qq = tt$, and since we let $q = up$, it will be

$$pp = \frac{t}{1 + uu} \quad \text{and} \quad d \cdot \frac{p}{q} = du = \frac{pdq - qdp}{pp},$$

and hence

$$pdq - qdp = ppdu = \frac{ttdu}{1 + uu},$$

and when these values are substituted, because

$$q - \pi p = \frac{vtt}{p(1 + vu)} \quad \text{and} \quad dp + \pi dq = \frac{tdt - (vttdu) : (1 + uu)}{p(1 + uv)},$$

it will be

$$0 = -\frac{(du(1+vv) - dv(1+uu))(1+tt)}{(1+uv)^2} - \frac{vtt(t(1+uu)dt - vttdu)}{pp(1+uu)(1+uv)^2}$$

or

$$(1+tt)(du(1+vv) - dv(1+uu)) + vt((1+uu)dt - vtdu) = 0,$$

which can then be reduced to this form:

$$du((1+vv)(1+tt) - vvtt) - dv(1+tt)(1+uu) + vtdt(1+uu) = 0$$

or this even more neat one:

$$\frac{du}{1+uu}(1+vv+tt) - dv(1+tt) + vtdt = 0.$$

Now let us put $v = w\sqrt{1+tt}$, and it will be

$$dw = \frac{dv(1+tt) - vtdt}{(1+tt)^{\frac{3}{2}}}$$

that is, it will be

$$dv(1+tt) - vtdt = (1+tt)^{\frac{3}{2}}dw;$$

then indeed it will be

$$1 + tt + vv = (1+tt)(1+ww),$$

and with these values substituted our equation itself thus becomes

$$\frac{du}{1+uu}(1+tt)(1+ww) - (1+tt)^{\frac{3}{2}}dw = 0,$$

hence by separating we obtain

$$\frac{du}{1+uu} = \frac{dw\sqrt{1+tt}}{1+ww},$$

and consequently

$$\frac{dw}{1+ww} = \frac{du}{(1+uu)\sqrt{1+tt}},$$

which equation can always be intergated whenever t is a function of u or whenever $pp + qq$ is a function of $\frac{q}{p}$ or q a function of p.

17. It will further turn out, so that q is a function of p, first if z and y are thus determined by x and another new variable ω, so that it would be $y = Ax$ and $z = Bx$, with A and B being some functions of ω. Therefore since we have put $dz = pdx + qdy$, it will be

$$Bdx + xdB = pdx + qAdx + qxdA,$$

where the terms involving the differential should be compared separately, whence it will be $p = B - Aq$; and comparing separately the terms containing the quantity x it will be $q = \frac{dB}{dA}$ and hence $p = \frac{BdA - AdB}{dA}$ And thus p and q are functions of ω and hence $tt = pp + qq$ and $u = \frac{p}{q}$ will be functions of the single quantity ω and $\sqrt{1+tt}$ will be a function of u. Because of this, the equation found above for the shortest path admits integration. Moreover, in this case, namely in which $y = Ax$ and $z = Bx$, a conical surface follows constructed upon any base.

18. The equation given above would then be integrable when we put $y = Ax + C$ and $z = Bx + D$; for then it will be

$$dz = pdx + qdy = Bdx + xdB + dD$$

and because $dy = Adx + xdA + dC$, it will also be

$$dz = pdx + qdy = pdx + Aqdx + xqdA + qdC$$

and hence, comparing the members containing the quantity x with each other, and then indeed those which are endowed with the differential dx, it will be

$$B = p + Aq \quad \text{and} \quad dB = qdA,$$

then

$$q = \frac{dB}{dA} \quad \text{and} \quad p = \frac{BdA - AdB}{dA}.$$

Besides these indeed it should be $dD = qdC = \frac{dBdC}{dA}$, or the functions A, B, C, D should thus be compared that $dAdD = dBdC$. If this occurs, p and q will again be function of the single variable ω and hence $\sqrt{1+tt}$ will also be a function of u, in which case too it is possible to define a shortest path. This case seems indeed to complete all the planar surfaces which can be explained in the plane.